Optimally swimming mechanisms at low Reynolds number

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Joint works with
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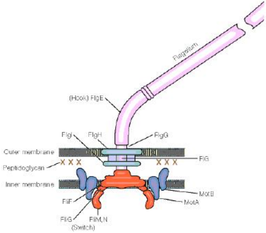
Aims and questions

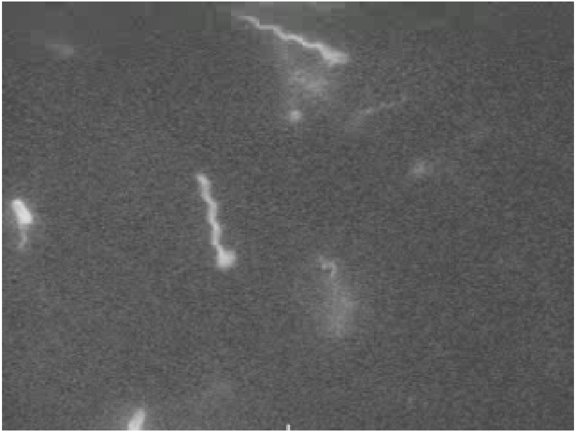
Study swimming micro-mechanisms

Swimming at low Reynolds number

- What are the best mechanisms ?
- Which shape?
- Which propulsion mechanism ?
- Self propulsion vs external propulsion...

E. Coli


Flagellum
(Hook) FlgE
FlgI
FlgH
FlgG
Outer membrane
FlG
Peptidoglycan
Inner membrane
MotB
MotA
FliF
FliG
FliM, N
(Switch)



Euglena

ESPCI (2005)

- Head: Red blood cell
- Tail: magnetic particles linked with DNA

Dreyfus et al, Nature 437(7060), 862–865, 2005

What is swimming?

Definition: “Ability to move inside or on water with appropriate periodic (stroke) movements and without external forces

Control problem: Given a deformable body, is it possible to find an internal force law that produces a periodic shape deformation that induces a displacement through the fluid reaction?

Optimal control problem: If possible, how to swim the most efficiently possible?

Low Reynolds number

$$\left[\begin{array}{l} \rho\left(\frac{\partial u}{\partial t} + (u\cdot\nabla)u\right) - \nu\Delta u + \nabla p = f, \\ \text{div} u = 0 \end{array}\right.$$

For a bacterium $L \sim 1\mu m$, $U \sim 1\mu m/s$ and

$$Re = \frac{\rho UL}{\nu} \sim 10^{-6}$$

Right model: Stokes equations

$$\left[\begin{array}{l} -\nu\Delta u + \nabla p = f, \\ \text{div} u = 0 \end{array}\right.$$

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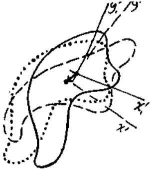
Low Reynolds situations $= \frac{\rho UL}{\nu}$

- In water, small sizes and velocities $U, L \ll 1$ (typical biological flows)
- At human size, flows of viscous fluids $\nu \gg 1$, (honey, silicon, etc.)
- Extremely small velocities $U \lll 1$ and/or extremely viscous fluids (e.g.: glaciers)

Stokes equations



$$\begin{cases} -\nu \Delta U + \nabla P = 0 \\ \text{div} U = 0 \\ \sigma n = f \text{ on the swimmer} \\ U = U_S \text{ on the swimmer (non slip)} \end{cases}$$

- ν viscosity
- U velocity
- P pressure
- $\sigma = \nu(\nabla U + (\nabla U)^T) - P\,Id$ Cauchy stress tensor.
- f force density on the surface of the swimmer.

Stokes equations are linear

$$\Rightarrow f = \mathcal{L}_{(\xi,p)} U_S$$

Low Reynolds number flows

$$Re = \frac{\rho U L}{\nu} \ll 1$$

(Film: G. I. Taylor)

Reversibility

(Film: G. Blanchard, S. Calisti, S. Calvet, P. Fourment, C. Gluza, R. Leblanc, M. Quillas-Saavedra)

The scallop theorem

Obstruction:[Purcell]

In Stokes regime, a reciprocal shape change induces no global motion

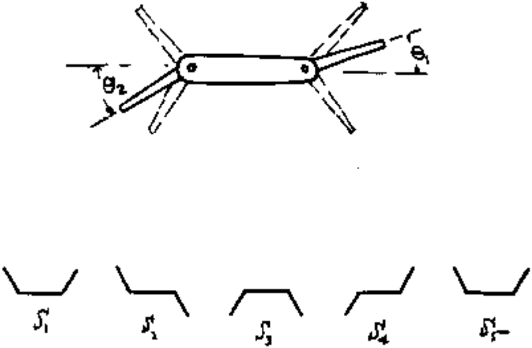
(Film: G. I. Taylor)

Summary

- Micro-swimming $\Rightarrow$ $Re \sim 0$
- Stokes equations for the fluid (linear)
- Flows are reversible (Scallop theorem)

The 3-link swimmer (Purcell)

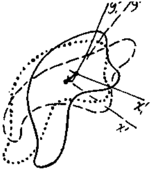
Edward Mills Purcell
(1912 - 1997)



Setting of the problem

- Low Reynolds number swimmers
- Shape induced swimming
- Self propulsion
- Inertia is negligible

$$\begin{cases} F_{tot} = 0, \\ T_{tot} = 0 \end{cases}$$

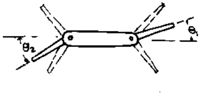


[Purcell]

Mathematical modelling

- The swimmer is characterized by its shape ξ and its position p

  Example: Purcell 3-link swimmer



  - $\xi = (\theta_1, \theta_2)$,
  - p =position and orientation.

- The swimmer can change shape $\Rightarrow \xi(t)$ pushing the fluid
- The fluid reacts obeying Stokes equations pulling the swimmer $\Rightarrow p(t)$

Mathematical modelling (cont'd)
The dynamics

- ξ = the shape, p = the position
- U_S is linear in $\dot{\xi}$ and $\dot{p}$
- $\Rightarrow$ f is linear in $\dot{\xi}$ and $\dot{p}$
- $\Rightarrow$ F_{tot} and T_{tot} are linear in $\dot{\xi}$ and $\dot{p}$:

$$\begin{pmatrix} F_{tot} \\ T_{tot} \end{pmatrix} = A(\xi, p)\dot{p} + B(\xi, p)\dot{\xi}$$

$$\dot{p} = V(\xi, p)\dot{\xi}$$

Mathematical modelling (cont'd)
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Proof of the scallop theorem

The scallop has only one degree of freedom ξ. The system becomes

$$\begin{array}{rcl} \dot{\xi} & = & \alpha(t) \\ \dot{p} & = & V(\xi)\dot{\xi} \end{array}$$

and $p = \int^{\xi} V(y)dy =: W(\xi)$

If ξ is periodic, so is p...

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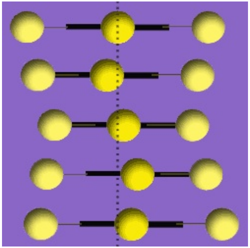
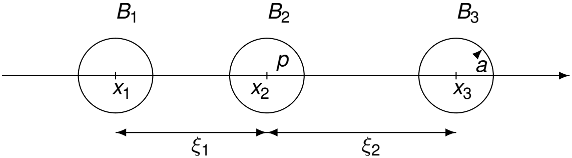
The 3-sphere swimmer (Najafi & Golestanian)



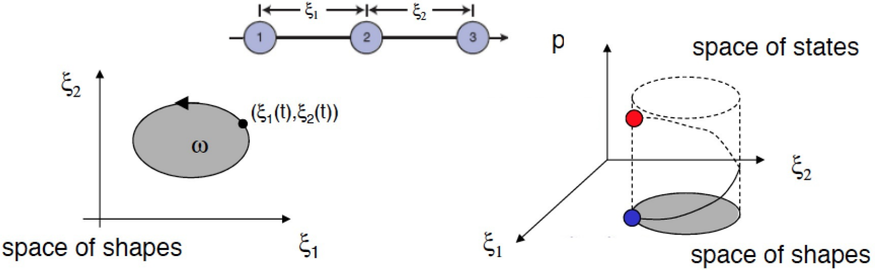


- (ξ_1, ξ_2) lengths or the arms, p position of central ball
- By changing ξ_1 and ξ_2, the spheres impose forces f_1, f_2, f_3 to the fluid with $f_1 + f_2 + f_3 = 0$
- 3 variables ξ_1, ξ_2, p and 2 control parameters
- Velocities (and forces) are linear in $\dot{\xi}_1$, $\dot{\xi}_2$, $\dot{p}$

$$\dot{p} = V_1(\xi, \not{p})\dot{\xi}_1 + V_2(\xi, \not{p})\dot{\xi}_2$$

Dynamical system

$$\dot{p} = V_1(\xi)\dot{\xi}_1 + V_2(\xi)\dot{\xi}_2$$



$$\Delta p = \int_0^T \begin{pmatrix} V_1(\xi) \\ V_2(\xi) \end{pmatrix} \cdot \frac{d\xi}{dt} = \int_\omega \left(\frac{\partial V_2}{\partial \xi_1} - \frac{\partial V_1}{\partial \xi_2} \right) d\sigma,$$

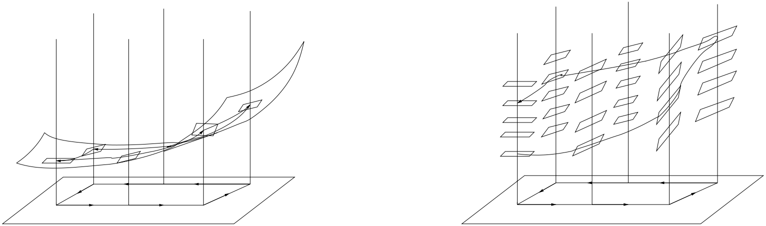
(due to Stokes' theorem)

Holonomic vs nonholonomic constraints...

$$\dot{p} = V_1(\xi)\dot{\xi}_1 + V_2(\xi)\dot{\xi}_2$$

$$\frac{d}{dt}\underbrace{\begin{pmatrix}\xi_1\\ \xi_2\\ p\end{pmatrix}}_{\text{state}} = \dot{\xi}_1\begin{pmatrix}1\\ 0\\ V_1(\xi)\end{pmatrix} + \dot{\xi}_2\begin{pmatrix}0\\ 1\\ V_2(\xi)\end{pmatrix} = \alpha_1(t)g_1(\xi) + \alpha_2(t)g_2(\xi)$$

At (ξ_1, ξ_2, p), the trajectory is tangent to the plane $(g_1(\xi), g_2(\xi))$



$p = W(\xi_1, \xi_2)$ $\qquad$ $\dot{p} = V_1(\xi)\dot{\xi}_1 + V_2(\xi)\dot{\xi}_2$

equivalent if $V = \nabla_\xi W$ or $\operatorname{curl} V = \frac{\partial V_2}{\partial \xi_1} - \frac{\partial V_1}{\partial \xi_2} = 0$

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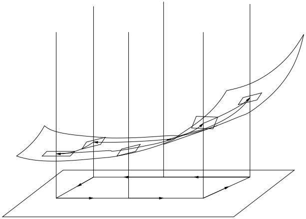

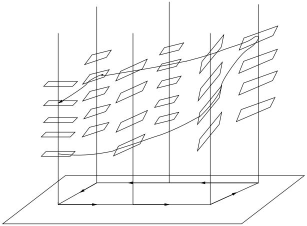

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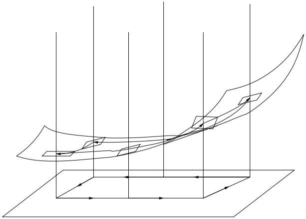

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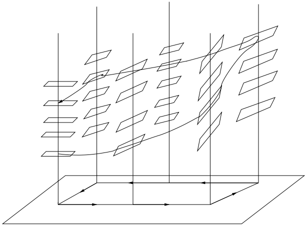

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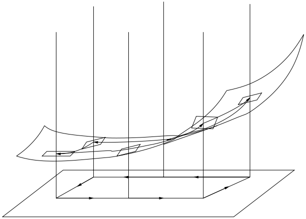

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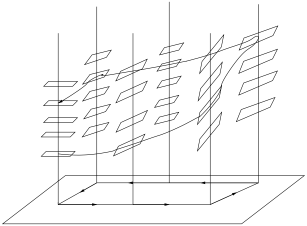

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The dynamical system

$$\underbrace{\frac{d}{dt}\begin{pmatrix} \xi_1 \\ \xi_2 \\ p \end{pmatrix}}_{\text{state} = X} = \dot{\xi}_1 \begin{pmatrix} 1 \\ 0 \\ V_1(\xi) \end{pmatrix} + \dot{\xi}_2 \begin{pmatrix} 0 \\ 1 \\ V_2(\xi) \end{pmatrix} = \alpha_1(t) g_1(\xi) + \alpha_2(t) g_2(\xi)$$

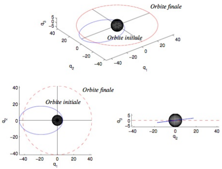
$$\dot{X} = \alpha_1(t) g_1(X) + \alpha_2(t) g_2(X)$$

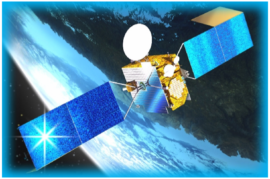
- $X =$ state (3D vector)
- α_1 and α_2 are the controls (rate of shape changes)
- $g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}^3$ vectorfields

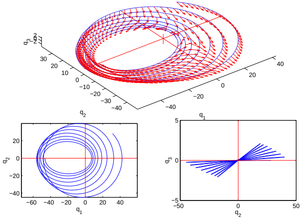
$\Rightarrow$ Control Theory

(Is it possible to drive a system from an initial point to a final point with an appropriate control?)

Controllability







Control theory

Is it possible to drive a system from an initial point to a final point with an appropriate control?

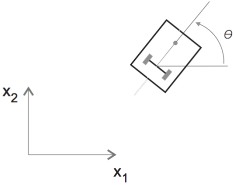
Take a system of the form

$$\begin{cases} \dot{X} = f(X, \alpha) \\ X(0) = X_0 \end{cases}$$

X = state, α = control(s)

- For each control $\alpha(t)$ one can uniquely solve the equation on $[0, T]$ and the system arrives at $X(T)$.
- Question: Is it possible to describe the attainable set $\{X(T)\}$ when $\alpha(t)$ varies?
- The system is locally controllable if one can reach any point in a neighborhood of X_0 starting from X_0 with a suitable control
- The system is globally controllable if one can reach any point in the state space starting from X_0 with a suitable control

An example: A model car



Position (x, y) angle θ
Controls $\alpha_1 = \textit{velocity}$, $\alpha_2 = \dot{\theta}$

$$\begin{cases} \frac{dx}{dt}(t) = \alpha_1(t)\cos(\theta(t)) \\ \frac{dy}{dt}(t) = \alpha_1(t)\sin(\theta(t)) \\ \frac{d\theta}{dt}(t) = \alpha_2(t) \end{cases} \Rightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \\ \theta \end{pmatrix} = \alpha_1(t)\begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} + \alpha_2(t)\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

Lie brackets

$$\dot{X} = \sum_{i=1}^{m} \alpha_i(t) g_i(X), \;\; X \in \mathbb{R}^n \text{ and } m < n$$

We start from $X(0) = X_0$,

- Take $\alpha_1 = 1$ and $\alpha_j = 0$ for $j \neq 1$ during a time ε

$$X(\varepsilon) = X_0 + \varepsilon g_1(X_0) + O(\varepsilon^2)$$

- Similarly taking $\alpha_i = 1$ and $\alpha_j = 0$ for $j \neq i$ during a time ε

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- Take

$$\begin{aligned} (\alpha_1, \alpha_2) &= (1, 0) && \text{on} && [0, \varepsilon[, \\ (\alpha_1, \alpha_2) &= (0, 1) && \text{on} && [\varepsilon, 2\varepsilon[, \\ (\alpha_1, \alpha_2) &= (-1, 0) && \text{on} && [2\varepsilon, 3\varepsilon[, \\ (\alpha_1, \alpha_2) &= (0, -1) && \text{on} && [3\varepsilon, 4\varepsilon[. \end{aligned}$$

then $X(4\varepsilon) = X_0 + \varepsilon^2 [g_1, g_2](X_0) + O(\varepsilon^3)$, where

$$[g_1, g_2] = (g_1 \cdot \nabla) g_2 - (g_2 \cdot \nabla) g_1$$

is the Lie bracket between g_1 and g_2 at X_0.

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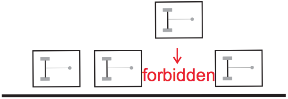
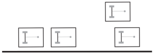
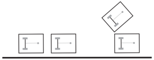
Lie brackets and car parking

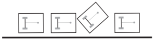




1. motion forward



2. rotation counterclockwise



3. motion backward

4. rotation clockwise

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \alpha_1(t) \begin{pmatrix} \cos(\theta) \\ \sin(\theta) \\ 0 \end{pmatrix} + \alpha_2(t) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

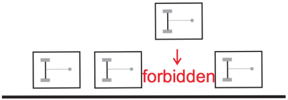
$$= \alpha_1(t) g_1(x, y, \theta) + \alpha_2(t) g_2(x, y, \theta)$$

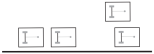
$$[g_1, g_2] = (g_1 \cdot \nabla) g_2 - (g_2 \cdot \nabla) g_1$$

$$= 0 - \frac{\partial}{\partial \theta} g_1$$

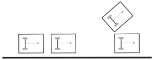
$$= \begin{pmatrix} \sin(\theta) \\ -\cos(\theta) \\ 0 \end{pmatrix}$$

Lie brackets and car parking

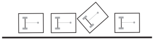




1. motion forward



2. rotation counterclockwise



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Chow's theorem

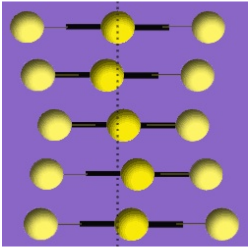
- Any Lie bracket generates a possibly new direction.
- One can iterate Lie brackets to generate even more new directions. E.g. $[g_1, [g_1, g_2]]$, $[g_3, [g_1, [g_1, g_2]]]$, etc.

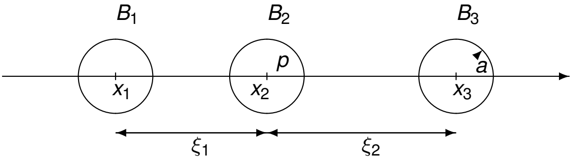
We call $Lie(g_1, \cdots, g_n)(X_0)$ the Lie algebra generated by the vectorfields $(g_1, \cdots, g_n)$ at X_0 and iterated Lie brackets.

Chow's theorem (1937) :

- If $dim(Lie(g_1, \cdots, g_n)(X_0) = dim(X)$, then the system is locally controlable at X_0. (One can reach any final point X_{final} in a neighborhood of the initial point X_0 in any time).
- If $dim(Lie(g_1, \cdots, g_n)(X_0) = dim(X)$ for every initial point X_0 then the system is globally controlable (one can reach any final point X_{final} from any initial point X_0 in any time).

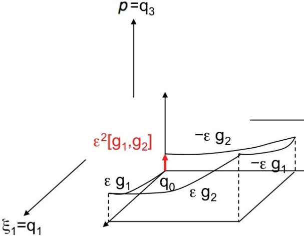
Back to N.-G. swimmer





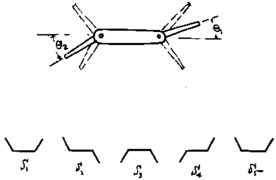
$$\underbrace{\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\\ p\end{pmatrix}}_{\text{state} = X} = \alpha_1(t)\underbrace{\begin{pmatrix}1\\ 0\\ V_1(\xi)\end{pmatrix}}_{g_1(X)} + \alpha_2(t)\underbrace{\begin{pmatrix}0\\ 1\\ V_2(\xi)\end{pmatrix}}_{g_2(X)}$$

$$[g_1, g_2] = (g_1 \cdot \nabla)g_2 - (g_2 \cdot \nabla)g_1 = \begin{pmatrix}0\\ 0\\ \frac{\partial V_2}{\partial \xi_1} - \frac{\partial V_1}{\partial \xi_2}\end{pmatrix}$$

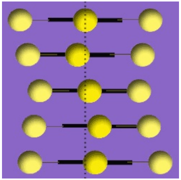


Conclusion: The system is controlable (the NG swimmer can swim) iff $det(g_1, g_2, [g_1, g_2]) = \frac{\partial V_2}{\partial \xi_1} - \frac{\partial V_1}{\partial \xi_2} \neq 0$.

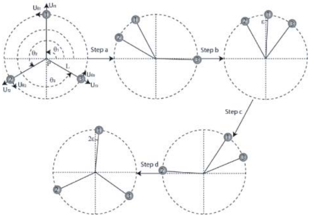
Other examples



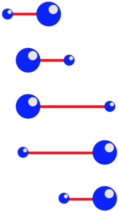
[3-link swimmer (E. M. Purcell)]



[3-sphere swimmer (Najafi & Golestanian)]



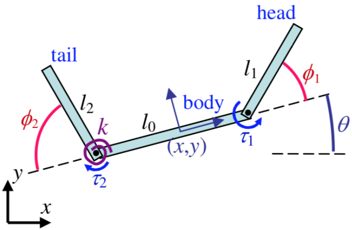
[Purcell rotator (R. Dreyfus et al)]



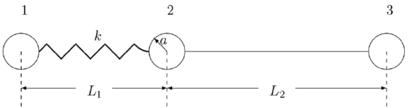
[Pushmepullyou (J. E. Avron)]

Ingredient: Looping in the shape space to produce a Lie bracket displacement...

Swimming with only one active arm



Passov & Or, EPJE 2012



Montino & DeSimone EPJE 2015

A control theorem

$$\frac{d}{dt}\begin{pmatrix}\xi_1\\ \xi_2\\ p\end{pmatrix} = \alpha_1(t)g_1(\xi) + \alpha_2(t)g_2(\xi).$$

Theorem (DeSimone, Lefebvre, A.)

The 3-sphere swimmer is globally controllable.

From any state (ξ_1^i, ξ_2^i, p^i), one can reach any other state (ξ_1^f, ξ_2^f, p^f) with suitable force laws $(f_j(t))_j$ such that $\sum_j f_j(t) = 0$ (or equivalently fonctions $\alpha_j(t)$).

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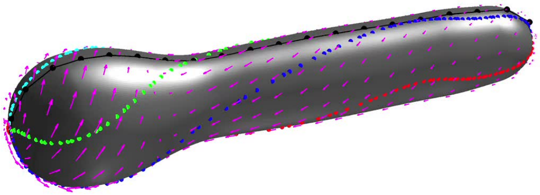
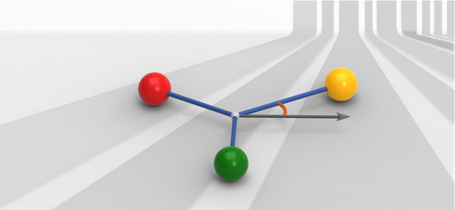
Euglena

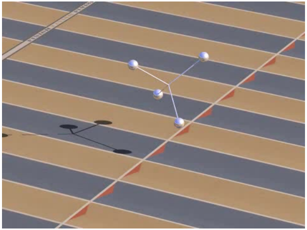


Arroyo et al, PNAS 109(44) 2012

Other controllable systems (DeSimone, Lefebvre, Merlet, A.)



3 controls, 3 first order Lie brackets



4 controls, 6 first order Lie brackets

Temporary conclusions

One possible way to overcome the scallop Theorem is

- to loop in the shape space...
- in order to generate Lie brackets...
- that produce new directions for the dynamical system.
- The displacement is proportional to the area (measured with curl V) enclosed by the loop

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Optimal swimming at low Re

Optimal parking problem?

Optimal Swimming at low Re

Goal: Find optimal swimming strategies.

We use Lighthill efficiency:

- A stroke $\xi(t)$ produces a displacement Δp
- Compare the energy expanded by the stroke to the one needed to pull the swimmer by Δp during the same time T.

Energy expanded = $\int_0^T f \cdot v \, dt$

Energy needed to pull the swimmer = Cte $T \left(\frac{\Delta p}{T}\right)^2$.

$$\text{Efficiency}^{-1} = \frac{\int_0^T f \cdot v \, dt}{Cte\, T \left(\frac{\Delta p}{T}\right)^2}$$

Maximizing the efficiency means finding the stroke(s) that produce a given displacement Δp during fixed time $T (= 2\pi)$ and which miminum energy

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(Lighthill) Efficiency

Find a (the?) stroke that produces the displacement Δp at least cost

- Forces $(f_i)_i$ depend linearly of the velocities $(v_i)_i$
- Velocities depend linearly of $(\dot{\xi}, \dot{p})$
- $\dot{p}$ is linear in $\dot{\xi}$

$$\int_0^{2\pi} f(t) \cdot v(t)\, dt = \int_0^{2\pi} (G(\xi)\dot{\xi}(t), \dot{\xi}(t))\, dt$$

where $G = (g_{ij})$ is a (dissipation) metric

- Optimal strokes can be interpreted as geodesics (in a subRiemannian space). They are closed loops in the shape space.

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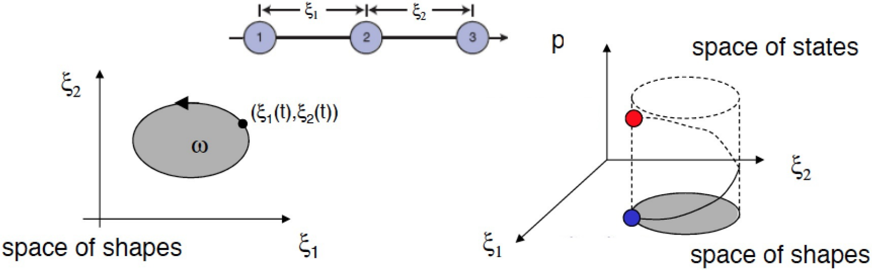
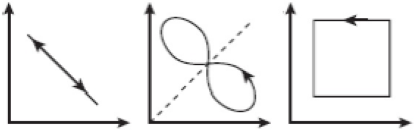
subRiemannian geodesics



But $\Delta p = \int_0^T V(\xi) \cdot \frac{d\xi}{dt} = \int_\omega \text{curl } V \, d\sigma$ is fixed
and $E = \int_0^T G(\xi)\dot{\xi} \cdot \dot{\xi} \, dt \to \min$

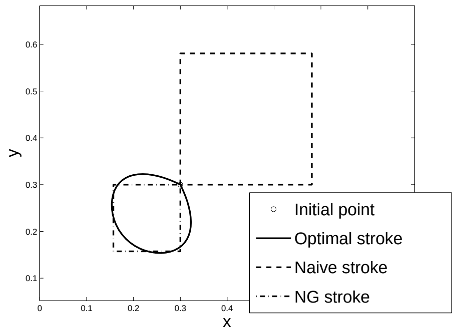
Isoperimetric problem

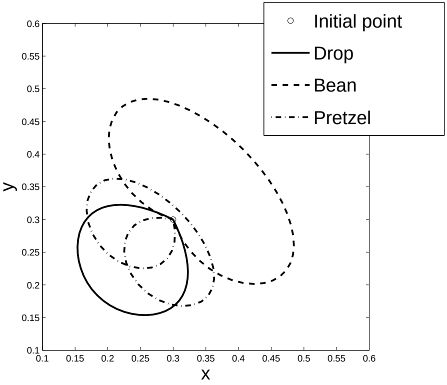


$\Delta p = 0,$ $\Delta p = 0,$ $\Delta p \neq 0?$

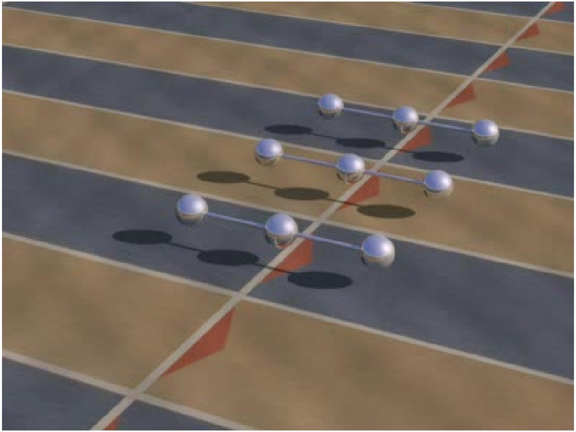
subRiemannian geodesics

- Discretization (Approximation and/or Finite elements/BEM techniques)
- Optimization (Handmade solving the geodesic equation and/or use Trilinos optimization toolbox)

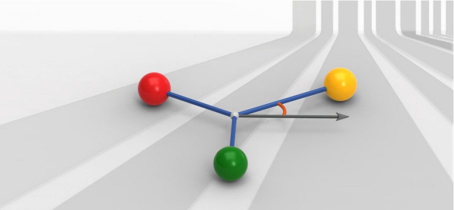




Optimal NG Swimmer



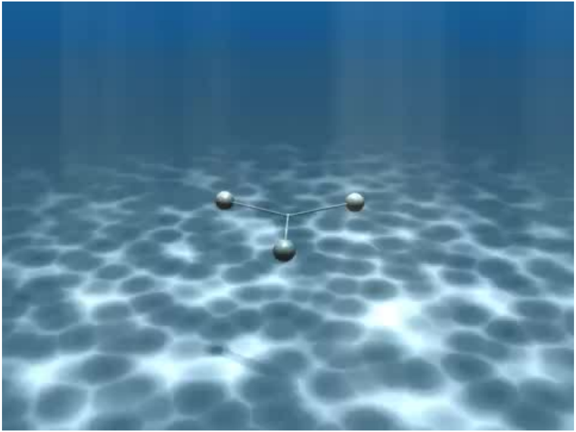
Back to the Plane swimmer



- 3 arms making 120°one to another
- 3 controls (extensible arms)
- 3 controllable changes of position (2 translation + 1 rotation)

What do optimal gaits look like?

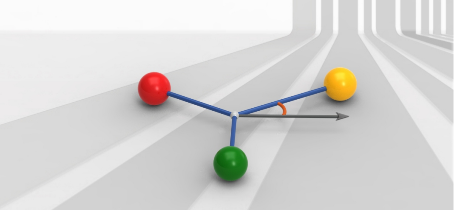
Optimal Plane Swimmer (large strokes)



Alouges et al, DCDS - B,18(5),1189–1215, 2013

Difficult to analyze

Optimal Plane Swimmer (small strokes)



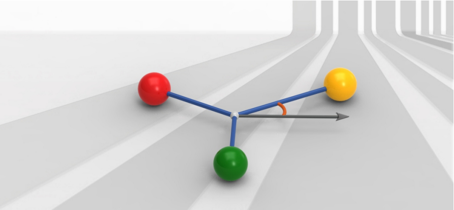
Position $p = (x, y, \theta)$ and Shape $\xi = (\xi_1, \xi_2, \xi_3)$

The optimal stroke problem

Find $\min_\xi \int_0^{2\pi} G(\xi)\dot{\xi}(t) \cdot \dot{\xi}(t)\, dt$ under the constraints

- ξ is 2π periodic
- $\dot{p} = V(\xi, p)\dot{\xi}$, and $\int_0^{2\pi} \dot{p}\, dt = \Delta p$ is given

Optimal Plane Swimmer (small strokes)



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Optimal Plane Swimmer

Difficulties

- $V(\xi, p)$ and $G(\xi)$ are not explicit
- The Euler-Lagrange equations of the optimization problem are highly nonlinear

Invariance and symmetries

- V does not depend on x, y
- V does depend on θ in a explicit way

$$\begin{cases} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} V_x(\xi,\theta) \\ V_y(\xi,\theta) \end{pmatrix} \dot{\xi} = R_\theta \begin{pmatrix} V_x(\xi,0) \\ V_y(\xi,0) \end{pmatrix} \dot{\xi} \\ \dot{\theta} = V_\theta(\xi)\dot{\xi} \end{cases}$$

- Consider small deformations near the symmetric shape $\xi_0 = (l_0, l_0, l_0)$ and linearize everything

$$V_i(\xi) \sim V_i(\xi_0) + \nabla_\xi V_i(\xi_0) \cdot \delta\xi$$

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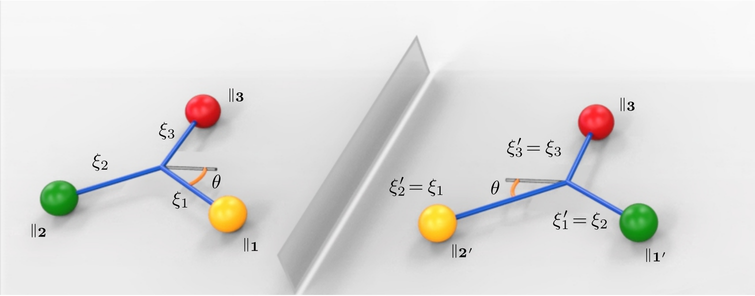
Symmetries



Expansions

Assume $|\delta\xi|, |\dot{\delta\xi}| \sim \epsilon \ll 1$

The symmetric shape ξ_0 furthermore implies that

- $V_\theta(\xi_0) = 0$
- To leading order, one has $\dot{\theta} = \nabla_\xi V_\theta(\xi_0)\delta\xi\dot{\delta\xi}$ Assuming $\theta(0) = 0$, one deduces $\theta(t) = O(\epsilon^2)$
- And finally

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} V_x(\xi_0) + \nabla_\xi V_x(\xi_0)\delta\xi \\ V_y(\xi_0) + \nabla_\xi V_y(\xi_0)\delta\xi \end{pmatrix} \dot{\delta\xi} + O(\epsilon^3)$$

- Integrating over a period gives

$$\begin{cases} \Delta x = \int_0^{2\pi} (\nabla_\xi V_x)_{\text{skew}}(\xi_0)\delta\xi \cdot \dot{\delta\xi}\, dt + O(\epsilon^3) \\ \Delta y = \int_0^{2\pi} (\nabla_\xi V_y)_{\text{skew}}(\xi_0)\delta\xi \cdot \dot{\delta\xi}\, dt + O(\epsilon^3) \\ \Delta \theta = \int_0^{2\pi} (\nabla_\xi V_\theta)_{\text{skew}}(\xi_0)\delta\xi \cdot \dot{\delta\xi}\, dt + O(\epsilon^3) \end{cases}$$

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Symmetry arguments show that $\exists \alpha, \gamma$ s.t.

$$(\nabla_\xi V_x)_{\text{skew}}(\xi_0) = \begin{pmatrix} 0 & \alpha & \alpha \\ -\alpha & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix}, \quad (\nabla_\xi V_y)_{\text{skew}}(\xi_0) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & \alpha & -\alpha \\ -\alpha & 0 & -2\alpha \\ \alpha & 2\alpha & 0 \end{pmatrix},$$

$$(\nabla_\xi V_\theta)_{\text{skew}}(\xi_0) = \begin{pmatrix} 0 & \gamma & -\gamma \\ -\gamma & 0 & \gamma \\ \gamma & -\gamma & 0 \end{pmatrix}$$

In other words

$$\begin{cases} \Delta x = \tau_x \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) \\ \Delta y = \tau_y \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) \\ \Delta \theta = \tau_\theta \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) \end{cases}$$

where $\tau_x = \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\tau_y = \frac{\alpha}{\sqrt{3}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\tau_\theta = \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Symmetry arguments show that $\exists \alpha, \gamma$ s.t.

$$(\nabla_\xi V_x)_{\text{skew}}(\xi_0) = \begin{pmatrix} 0 & \alpha & \alpha \\ -\alpha & 0 & 0 \\ -\alpha & 0 & 0 \end{pmatrix}, \quad (\nabla_\xi V_y)_{\text{skew}}(\xi_0) = \frac{1}{\sqrt{3}} \begin{pmatrix} 0 & \alpha & -\alpha \\ -\alpha & 0 & -2\alpha \\ \alpha & 2\alpha & 0 \end{pmatrix},$$

$$(\nabla_\xi V_\theta)_{\text{skew}}(\xi_0) = \begin{pmatrix} 0 & \gamma & -\gamma \\ -\gamma & 0 & \gamma \\ \gamma & -\gamma & 0 \end{pmatrix}$$

In other words

$$\begin{cases} \Delta x = \tau_x \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) \\ \Delta y = \tau_y \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) \\ \Delta \theta = \tau_\theta \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) \end{cases}$$

where $\tau_x = \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\tau_y = \frac{\alpha}{\sqrt{3}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\tau_\theta = \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Symmetry arguments show that $\exists \alpha, \gamma$ s.t.

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$$(\nabla_\xi V_\theta)_{\text{skew}}(\xi_0) = \begin{pmatrix} 0 & \gamma & -\gamma \\ -\gamma & 0 & \gamma \\ \gamma & -\gamma & 0 \end{pmatrix}$$

In other words

$$\begin{cases} \Delta x = \tau_x \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) = \text{Flux of } \tau_x \text{ through the loop} \\ \Delta y = \tau_y \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) = \text{Flux of } \tau_y \text{ through the loop} \\ \Delta \theta = \tau_\theta \cdot \int_0^{2\pi} \delta\xi \wedge \dot{\delta\xi}\, dt + O(\epsilon^3) = \text{Flux of } \tau_\theta \text{ through the loop} \end{cases}$$

where $\tau_x = \alpha \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$, $\tau_y = \frac{\alpha}{\sqrt{3}} \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$ and $\tau_\theta = \gamma \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

Optimal Plane Swimmer (small strokes) (DiFratta, A.)

Similarly, symmetry reasons show that

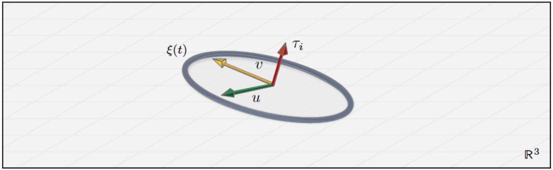
$$G(\xi_0) = \begin{pmatrix} \kappa & \rho & \rho \\ \rho & \kappa & \rho \\ \rho & \rho & \kappa \end{pmatrix}$$

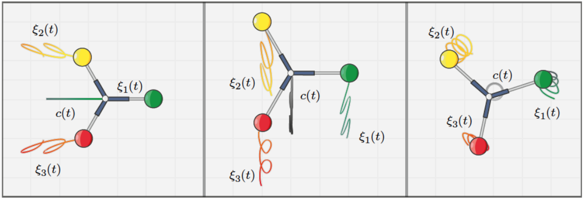
- In the regime of small strokes near the shape (ξ_0, ξ_0, ξ_0), an optimal stroke is a planar ellipse $\mathcal{E}$ (in the 3d shape space).
- The respective displacements in x, y, θ of the swimmer after one stroke are obtained by computing the flux through $\mathcal{E}$ of the vectors

$$\tau_x = \alpha(0, -1, 1), \ \tau_y = \frac{\alpha}{\sqrt{3}}(-2, 1, 1), \ \tau_\theta = \gamma(1, 1, 1)$$

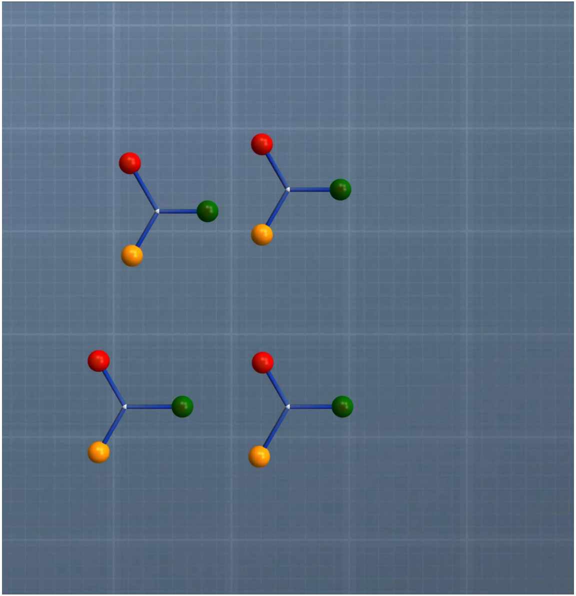
Complete characterization of the optimal (small) stroke that provides a prescribed displacement.

Optimal strokes

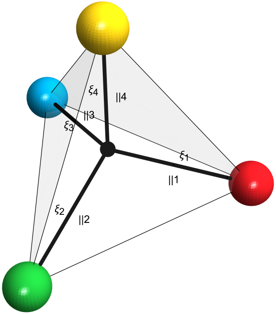




Optimal Plane Swimmer (small strokes)



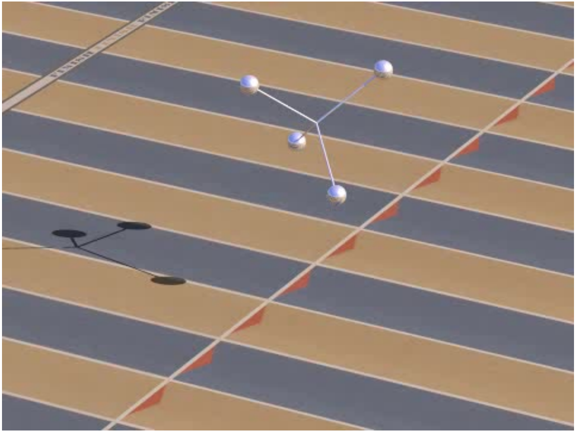
The 4 sphere swimmer



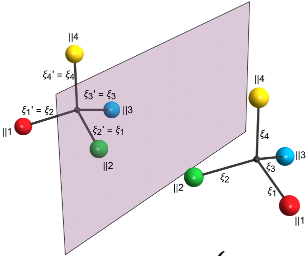
More involved:

- 4 controls
- 6 dimensions to control (3 translations, 3 rotations)

The 4 sphere swimmer



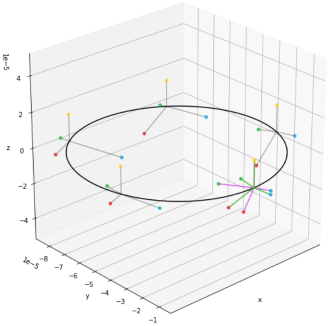
Same strategy...



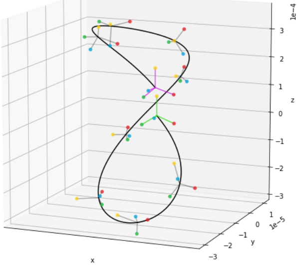
$$\begin{cases} \Delta x = \tau_x \cdot \int_0^{2\pi} \delta\xi \wedge \delta\dot{\xi}\, dt + O(\epsilon^3) \\ \Delta y = \tau_y \cdot \int_0^{2\pi} \delta\xi \wedge \delta\dot{\xi}\, dt + O(\epsilon^3) \\ \Delta z = \tau_z \cdot \int_0^{2\pi} \delta\xi \wedge \delta\dot{\xi}\, dt + O(\epsilon^3) \end{cases} \quad \begin{cases} \Delta R_1 = \tau_1 \cdot \int_0^{2\pi} \delta\xi \wedge \delta\dot{\xi}\, dt + O(\epsilon^3) \\ \Delta R_2 = \tau_2 \cdot \int_0^{2\pi} \delta\xi \wedge \delta\dot{\xi}\, dt + O(\epsilon^3) \\ \Delta R_3 = \tau_3 \cdot \int_0^{2\pi} \delta\xi \wedge \delta\dot{\xi}\, dt + O(\epsilon^3) \end{cases}$$

where $\tau_x, \tau_y, \tau_z, \tau_1, \tau_2, \tau_3$ are now... bivectors (explicit though)

2 optimal trajectories



Pure rotation along z



Translation along z and rotation along z

What changes?

- Complete characterization
- Optimal (small) strokes are described by

$$\xi(t) = \cos(t)u_1 + \sin(t)v_1 + \cos(2t)u_2 + \sin(2t)v_2$$

- Depending on the objective Δp one may have:
  - planar strokes ($u_2 = v_2 = 0$)
  - non planar strokes (general case)
  - infinitely many optimal strokes

Conclusion

- Complete understanding of optimal swimming (by shape deformation) for simple mechanisms
- Loop in a suitable space of deformations $\Rightarrow$ moves the system following a Lie bracket
- Optimal gaits are optimal loops
- In the regime of small deformations, it is possible to characterize them